J. Fluid Mech. (1979), *vol.* 91, *part* 1, *pp.* 167–180
Printed in Great Britain

Some effects of surface tension on steep water waves

By S. J. HOGAN

Department of Applied Mathematics and Theoretical Physics, University of Cambridge

(Received 19 April 1978 and in revised form 2 November 1978)

Surface tension provides a restoring force which cannot reasonably be ignored for water waves of short crest-to-crest length. Even for large wavelengths its presence precludes any sharp corner developing on the free surface. In this paper we begin an investigation of the effects of surface tension on steep water waves. The work of Longuet-Higgins (1975) is generalized to show how the integral properties of the wave train are affected. In particular it is shown that for pure capillary waves in deep water the mean fluxes of energy, mass and momentum are given by $3Tc$, $2T/c$ and $4T-V$ respectively, where c is the phase velocity, T the kinetic energy and V the potential energy.

Also the exact solution for the wave profile of deep-water pure capillary waves (Crapper 1957) is used to obtain wave profiles, all with the same mean level. This yields the unexpected result that the height of the wave crest above the mean level is *not* a monotonic function of wave steepness.

With subsequent papers this work will form one limiting case of the general problem of deep-water gravity–capillary waves.

1. Introduction

In this paper we begin an investigation of the effect of surface tension (capillarity) on steep periodic surface waves on water which is assumed to be inviscid, incompressible and irrotational. Previous work on steep *gravity* waves has been summarized in a paper by Longuet-Higgins (1977).

The inclusion of surface tension in the study of water waves has a long history. The first to consider gravity–capillary waves of finite amplitude was Harrison (1909). He quoted results of a third-order approximation scheme but did not analyse them in any great detail. Perhaps unaware of this work, Wilton (1915) carried out the expansion to fifth order and sketched profiles of short waves (less than 2 cm long). However, the most interesting part of Wilton's work arose because he noticed (as did Harrison) that, at certain known wavelengths, the perturbation expansion had a radius of convergence equal to zero. He reconsidered the problem, with a new expansion, and discovered that two waves could exist at the shortest of these wavelengths; he again gave sketches of their profiles. It was later found that this phenomenon results from the primary wave undergoing a resonant interaction with one of its harmonics. In 1920, Kamesvara Rav continued the analysis by including finite depth and considering circular waves which emanate from a point source (whereas previous work had been confined to plane, progressive waves); again, the expansions were found to fail at certain wavelengths for a given depth.

The question of the existence of plane progressive gravity–capillary waves was first considered by Sekerzh-Zenkovich (1956). Using the methods which Levi-Civita (1925) had adopted for pure gravity waves, he produced a proof for small amplitude waves. Beckert (1963) has also considered the same problem. We mention in passing the work of Slezkin (1935), who considered waves with surface tension providing the *only* restoring force. He gave a proof of their existence for sufficiently small wave steepness but missed an exact solution to the problem of finite amplitude waves, discovered in 1957 by Crapper. Pure capillary waves on water of finite depth have been investigated by Kinnersley (1976).

The rest of the work in this field has tended to concentrate on the failure of the perturbation expansion at certain wavelengths, as discovered by Wilton. Pierson & Fife (1961) tackled these singular cases by expanding about the relevant wavenumber and matching this solution to the solution they obtained by classical methods. Barakat & Houston (1968) continued with this approach in the case of water of finite depth and found that for *very* small depths the expansion could be regular. McGoldrick (1970) viewed the breakdown as a special case of interaction of two gravity–capillary waves with the correct initial conditions of propagation; Wilton had already shown that two waves could interact to give these ripples. Indeed the fact that the linear phase speed has a minimum does tend to confirm this conclusion, but McGoldrick showed how important the initial conditions are. Nayfeh (1970*a*, *b*, 1971, 1973) repeated Barakat & Houston's work but retained time in the analysis, instead of viewing the problem from a frame of reference moving with the waves. He concluded, by concentrating on different wavelengths, that the wavenumber expansion of Pierson & Fife was not uniformly valid. Also Lekoudis, Nayfeh & Saric (1977) analysed one of these singular wavelengths and included viscosity in their work, in an attempt to match theory and observation.

In § 2 of this paper, we prove some general relationships between various integral properties of gravity–capillary waves of arbitrary wavelength on water of finite depth and also quote the results for infinite depth. In § 3, Crapper's solution for the wave profile of pure capillary waves of finite amplitude is used to derive exact expressions for the kinetic and potential energies, together with the radiation stresses and energy flux, of these waves. Subsequent papers will present results both for the general case of gravity–capillary waves on water of infinite depth and for the particular case of the breakdown of the perturbation expansion at certain wavelengths.

2. The integral properties

This section introduces the notation and the co-ordinate system to be used throughout this work. We obtain some exact relationships for periodic gravity–capillary waves of finite amplitude on water of uniform depth. The methods are similar to those applied to pure gravity waves by Longuet-Higgins (1975). We also derive the results for the case of infinite depth.

Co-ordinate axes and definitions

Choose rectangular co-ordinates (x, y) with the x axis horizontal and the y axis vertically upwards. Let the equations of the free surface and of the bottom be $y = \eta$ and $y = -H$, respectively (H constant). The velocity (u, v) is assumed to be irrotational

($= \nabla\phi$) and periodic in x with finite wavelength λ. Viscosity is neglected and, as Wilton (1915) showed, this imposes restrictions on the range of λ to waves greater than 1 mm and less than 20 cm in length. Units are chosen such that the density has the value one.

Now choose axes such that the mean elevation $\bar{\eta}$, given by

$$\lambda\bar{\eta} = \int_0^\lambda \eta\,dx,$$

vanishes. We can do this because the wave is periodic. The origin is then fixed in the mean surface level and H becomes the mean depth.

Similarly, by choosing axes which move with the required horizontal velocity, we make the mean velocity $\bar{u}$, given by

$$\lambda\bar{u} = \int_0^\lambda u\,dx = [\phi]_{x=0}^{\lambda},$$

vanish at one level entirely within the fluid and so, since the flow is irrotational, at all such levels. This corresponds to the first definition of the phase velocity c, as given by Stokes (1847).

The mean wave momentum (or impulse) I, kinetic energy T and potential energy V (all per unit horizontal area) are defined by

$$I = \overline{\int_{-H}^{\eta} u\,dy}, \tag{2.1}$$

$$T = \overline{\int_{-H}^{\eta} \tfrac{1}{2}(u^2+v^2)\,dy}, \tag{2.2}$$

$$\begin{aligned} V &= \overline{\int_0^{\eta} gy\,dy} + \tau\overline{[(1+\eta'^2)^{\frac{1}{2}}-1]} \\ &= \overline{\{\tfrac{1}{2}g\eta^2+\tau[(1+\eta'^2)^{\frac{1}{2}}-1]\}} \\ &= V_g + V_\tau. \end{aligned} \tag{2.3}$$

Here an overbar denotes the average over one period (or wavelength, since the wave is progressive), g is the acceleration due to gravity, τ is the surface tension divided by the density and a prime denotes horizontal differentiation.

The mean flux of momentum per unit span is given by

$$W = \overline{\int_{-H}^{\eta} (p+u^2)\,dy} - \tau\,\overline{\cos\zeta},$$

where ζ is the angle the wave surface makes with the horizontal and p is the pressure.

The radiation stress per unit span, defined as the excess momentum flux due to the waves, is given by

$$S_{xx} = \overline{\int_{-H}^{\eta} (p+u^2)\,dy} + \tau\overline{(1-\cos\zeta)} - \tfrac{1}{2}gH^2. \tag{2.4}$$

Also, if (u_s, v_s) are the components of surface velocity, then the mean energy flux per unit span is defined by

$$F = \overline{\int_{-H}^{\eta} [p+\tfrac{1}{2}(u^2+v^2)+gy]\,u\,dy} + \tau\overline{[(1+\eta'^2)^{\frac{1}{2}}-1]\,u_s} + \tau\overline{\left[1-\frac{1}{(1+\eta'^2)^{\frac{1}{2}}}\right]u_s} - \tau\overline{\frac{\eta' v_s}{(1+\eta'^2)^{\frac{1}{2}}}}$$
$$= F_g + F_\tau. \tag{2.5}$$

Finally, the mass of fluid per unit horizontal area is given by

$$M = \overline{\int_{-H}^{\eta} dy} = H.$$

Momentum and kinetic energy

Using methods identical to those of Longuet-Higgins (1975), it is possible to show that

$$I = \frac{2T}{c}, \quad T = \frac{c}{2\lambda}\int \eta\, d\phi. \tag{2.6), (2.7}$$

These are the same as equations (B) and (2.6), respectively, of that paper.

Let Φ and Ψ denote the velocity potential and the stream function, respectively, of the motion relative to axes moving with the phase speed c. Now $x+iy$ is an analytic function of $\Phi+i\Psi$ so we can use the Cauchy–Riemann relations to obtain

$$\iint \frac{\partial y}{\partial \Psi}\, d\Phi\, d\Psi = \iint \frac{\partial x}{\partial \Phi}\, d\Phi\, d\Psi, \tag{2.8}$$

where the integral in Φ, Ψ space is over an area corresponding to one wavelength, from the bottom to the free surface in x, y space.

If we let $-Q$ be the mass flux in the steady flow, we have

$$-Q = \int_{-H}^{\eta} (u-c)\, dy = [\Psi]_{y=-H}^{\eta}.$$

On averaging both sides over one wavelength, this becomes

$$Q = cH - I. \tag{2.9}$$

The total head R' is given by

$$R' = p + \tfrac{1}{2}[(u-c)^2+v^2] + g(y+H),$$

and at the free surface this becomes

$$R = -(\tau\eta''/(1+\eta'^2)^{\frac{3}{2}}) + \tfrac{1}{2}q^2 + g(\eta+H), \tag{2.10}$$

where $R = R' - p_0$, p_0 being the atmospheric pressure. So

$$q = [2R + 2\tau\eta''/(1+\eta'^2)^{\frac{3}{2}} - 2g(\eta+H)]^{\frac{1}{2}}.$$

Also $$d\phi = d\Phi + c\,dx = -q\,ds + c\,dx = [-q(1+\eta'^2)^{\frac{1}{2}} + c]\,dx,$$

whence $$T = \frac{c}{2\lambda}\int_0^{\lambda} \eta\left\{c - (1+\eta'^2)^{\frac{1}{2}}\left[2R + \frac{2\tau\eta''}{(1+\eta'^2)^{\frac{3}{2}}} - 2g(H+\eta)\right]^{\frac{1}{2}}\right\} dx. \tag{2.11}$$

This is to be compared with equation (E) of Longuet-Higgins (1975), where $\tau = 0$ and a slight typographical error are both taken into account.

We can also obtain another expression for the kinetic energy which in turn enables us to state an expression for the phase speed c. From (2.8) we have

$$\int (H+\eta)\,d\Phi = \lambda \int d\Psi = -\lambda Q = \lambda(I-cH) = \lambda(2T/c - cH).$$

Hence
$$\int_0^\lambda q(H+\eta)(1+\eta'^2)^{\frac{1}{2}}\,dx = -\frac{2\lambda T}{c} + cH\int_0^\lambda dx,$$

so
$$T = \frac{c}{2\lambda}\int_0^\lambda \left\{cH - (H+\eta)(1+\eta'^2)^{\frac{1}{2}}\left[2R + \frac{2\tau\eta''}{(1+\eta'^2)^{\frac{3}{2}}} - 2g(H+\eta)\right]^{\frac{1}{2}}\right\} dx. \tag{2.12}$$

On equating (2.11) and (2.12) and rearranging, we obtain

$$c = \frac{1}{\lambda}\int_0^\lambda (1+\eta'^2)^{\frac{1}{2}}\left\{2R + \frac{2\tau\eta''}{(1+\eta'^2)^{\frac{3}{2}}} - 2g(H+\eta)\right\}^{\frac{1}{2}} dx. \tag{2.13}$$

Momentum flux and energy flux

The momentum flux and energy flux can be conveniently expressed in terms of known quantities of the fluid by the following methods.

Consider Bernoulli's equation in the form

$$[p + (u-c)^2] + (gy - c^2) + v^2 + (p + gy) = 2B, \tag{2.14}$$

where
$$B = R' - gh - \tfrac{1}{2}c^2,$$

and the equation of vertical momentum in the form

$$(y+H)\left[\frac{Dv}{Dt} + \frac{\partial}{\partial y}(p+gy)\right] = 0,$$

so that (2.14) becomes

$$[p + (u-c)^2 + (gy-c^2)] + \frac{D}{Dt}[(y+H)v] + \frac{\partial}{\partial y}[(y+H)(p+gy)] = 2B.$$

Then by integrating this equation over one wavelength and from the bottom to the free surface, in a manner identical to that of Longuet-Higgins (1975), and using (2.4) and (2.6), we obtain

$$S_{xx} = 4T + 2BH - \frac{1}{\lambda}\int_0^\lambda \left\{\tfrac{3}{2}g\eta^2 - \tau\left[\frac{\eta''(\eta+H)}{(1+\eta'^2)^{\frac{3}{2}}} - \frac{1}{(1+\eta'^2)^{\frac{1}{2}}} + 1\right]\right\} dx. \tag{2.15}$$

Note that
$$\cos\zeta = 1/(1+\eta'^2)^{\frac{1}{2}}.$$

Equation (2.15) can then be rewritten as

$$S_{xx} = 4T - 3V_g - V_\tau + 2BH \tag{2.16}$$

because
$$\frac{1}{\lambda}\int_0^\lambda \frac{\eta'' H}{(1+\eta'^2)^{\frac{3}{2}}}\,dx = \frac{H}{\lambda}\int_0^\lambda \frac{d(\eta')}{(1+\eta'^2)^{\frac{3}{2}}} = 0,$$

by periodicity, and

$$\int_0^\lambda \frac{\eta\eta''}{(1+\eta'^2)^{\frac{3}{2}}}\,dx = \int_0^\lambda \frac{dx}{(1+\eta'^2)^{\frac{1}{2}}} - \int_0^\lambda (1+\eta'^2)^{\frac{1}{2}}\,dx. \tag{2.17}$$

In (2.17) we use the fact that

$$\frac{d}{dx}\left[\frac{\eta'}{(1+\eta'^2)^{\frac{1}{2}}}\right] = \frac{\eta''}{(1+\eta'^2)^{\frac{3}{2}}}.$$

To obtain the energy flux let us rearrange (2.14) as

$$p + \tfrac{1}{2}(u^2+v^2) + gy = B + cu \tag{2.18}$$

so that, using (2.5), we obtain

$$F = \frac{1}{\lambda}\int_0^\lambda \int_{-H}^{\eta} (B+cu)\,u\,dy\,dx + F_\tau.$$

Thus

$$F = BI + c\overline{\int_{-H}^{\eta} u^2\,dy} + F_\tau \tag{2.19}$$

from (2.1).

Equation (2.18) can also be integrated; this gives

$$\overline{\int_{-H}^{\eta} (p+gy)\,dy} + T = BH + cI,$$

which, on using (2.6), yields

$$\overline{\int_{-H}^{\eta} (p+gy)\,dy} = T + BH. \tag{2.20}$$

Also

$$S_{xx} + V = \overline{\int_{-H}^{\eta} (p+gy)\,dy} + \overline{\int_{-H}^{\eta} u^2\,dy} + \tau\overline{\left[1 - \frac{1}{(1+\eta'^2)^{\frac{1}{2}}} + (1+\eta'^2)^{\frac{1}{2}} - 1\right]}$$

$$= T + BH + \overline{\int_{-H}^{\eta} u^2\,dy} + \tau\overline{\left[\frac{\eta'^2}{(1+\eta'^2)^{\frac{1}{2}}}\right]}.$$

So (2.19) becomes

$$F = BI + c\left\{S_{xx} + V - T - BH - \tau\overline{\left[\frac{\eta'^2}{(1+\eta'^2)^{\frac{1}{2}}}\right]}\right\} + F_\tau,$$

which can be written, using (2.16), (2.17) and the condition $(u_s - c)\,\eta' = v_s$, as

$$F = (3T - 2V_g)\,c + B(I + cH). \tag{2.21}$$

In the case of pure gravity waves, where $\tau = 0$, (2.16) and (2.21) reduce to equations (3.5) and (3.10) respectively of Longuet-Higgins (1975).

Infinite depth

Apart from the shallowest of water depths, short waves essentially propagate in water of infinite depth, and this case will now be briefly considered. Of the relations derived so far, (2.6) does not explicitly involve the depth and (2.11)–(2.13) will not be used directly. Consequently we turn our attention to (2.16) and (2.21). One term common to both these equations is BH. But this vanishes as $H \to \infty$ by a proof similar to that of Longuet-Higgins (1975). Consequently, we rewrite the mass flux I, momentum flux S_{xx} and energy flux F in the case of infinite depth as

$$I = 2T/c, \quad S_{xx} = 4T - 3V_g - V_\tau, \tag{2.22), (2.23}$$

$$F = (3T - 2V_g)\,c. \tag{2.24}$$

3. Pure capillary waves

We now deal with the case of pure capillary waves on water of infinite depth, that is, when the restoring force is derived from surface tension alone. The definitions in the previous section of the mean wave momentum I, kinetic energy T, potential energy V, radiation stress S_{xx} and energy flux F (from now on omitting 'per unit span' etc. for brevity) are slightly altered to become

$$I = \overline{\int_{-\infty}^{\eta} u\, dy}, \quad T = \overline{\int_{-\infty}^{\eta} \tfrac{1}{2}(u^2+v^2)\, dy}, \tag{3.1), (3.2}$$

$$V = \tau\overline{[(1+\eta'^2)^{\frac{1}{2}}-1]}, \tag{3.3}$$

$$S_{xx} = \overline{\int_{-\infty}^{\eta} (p+u^2)\, dy} + \tau\overline{\left[1-\frac{1}{(1+\eta'^2)^{\frac{1}{2}}}\right]}, \tag{3.4}$$

$$F = \overline{\int_{-\infty}^{\eta} [p+\tfrac{1}{2}(u^2+v^2)]\, u\, dy} + \tau\overline{\frac{\eta'(\eta' u_s - v_s)}{(1+\eta'^2)^{\frac{1}{2}}}}. \tag{3.5}$$

We obtain exact expressions for these quantities by making use of the exact solution for the surface profile of pure capillary waves (Crapper 1957). All these quantities are shown to be monotonic functions of wave steepness, in sharp contrast to the behaviour of pure gravity waves (see, for example, Longuet-Higgins 1975; Cokelet 1977).

Evaluation of potential and kinetic energies

The expressions for T and V, as given by (3.2) and (3.3), are readily evaluated by means of an exact solution for the surface profile of pure capillary waves (Crapper 1957). However, his results need slight modification owing to differences, in notation and in the co-ordinate axes, between his work and this (see figure 1). The equations of the surface in the present notation are

$$\frac{x-ct}{\lambda} = \frac{2A\sin 2\pi\alpha}{\pi(1+A^2+2A\cos 2\pi\alpha)} - \alpha \tag{3.6}$$

and

$$\frac{\eta}{\lambda} = \frac{2(1+A\cos 2\pi\alpha)}{\pi(1+A^2+2A\cos 2\pi\alpha)} - \frac{2}{\pi} - a_0, \tag{3.7}$$

where $y = \eta$ is the equation of the surface, $\alpha = \Phi/c\lambda$ varies from 0 to -1 over one wavelength, and a_0 is such that $\bar{\eta} = 0$. The quantity A is a strictly increasing function of the crest-to-trough wave height a for a given wavelength, and is related to the wave steepness a/λ by

$$a/\lambda = 4A/\pi(1-A^2);$$

hence

$$A = \frac{2\lambda}{\pi a}\left[\left(1+\frac{\pi^2 a^2}{4\lambda^2}\right)^{\frac{1}{2}} - 1\right]. \tag{3.8}$$

Now we know from Crapper's (1957) analysis that $0 \leqslant a/\lambda \leqslant 0{\cdot}729765$, so

$$0 \leqslant A \leqslant 0{\cdot}454670 = A_{\max}$$

and it is the quantity A, rather than a/λ, which plays the leading role in this section.

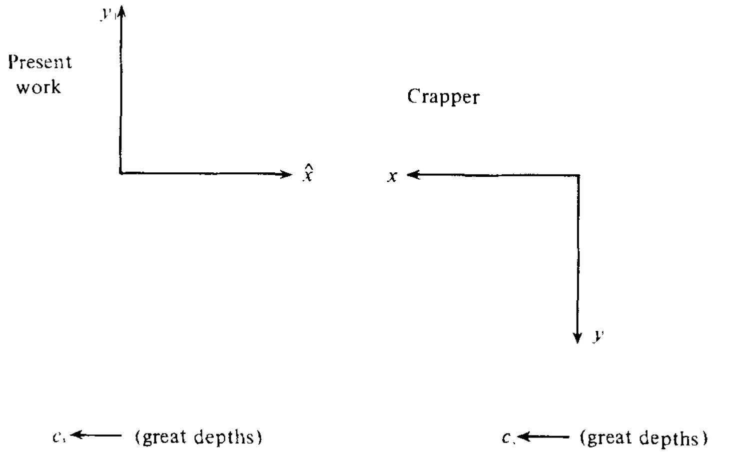

FIGURE 1

If we write $\hat{x}$ for $x-ct$ then we have, from (3.6) and (3.7),

$$\frac{1}{\lambda}\frac{d\hat{x}}{d\alpha} = \frac{-(A^4-6A^2+1+4A^2\cos^2 2\pi\alpha)}{(1+A^2+2A\cos 2\pi\alpha)^2} \tag{3.9}$$

and

$$\frac{1}{\lambda}\frac{d\eta}{d\alpha} = \frac{4A(1-A^2)\sin 2\pi\alpha}{(1+A^2+2A\cos 2\pi\alpha)^2}. \tag{3.10}$$

So, using (3.7) and (3.9),

$$\begin{aligned}\bar{\eta} \equiv \frac{1}{\lambda}\int_0^\lambda \eta\, dx &= \frac{-1}{\lambda}\int_0^1 \eta \frac{d\hat{x}}{d\alpha} d\alpha \\ &= -\frac{A\lambda}{\pi^2}\int_0^{2\pi} \frac{(A+\cos\theta)(A^4-6A^2+1+4A^2\cos^2\theta)}{(1+A^2+2A\cos\theta)^3} d\theta - a_0,\end{aligned}$$

where $\theta = 2\pi\alpha$. Now the integrand is the sum of terms of the form

$$\cos^m\theta/(1+A^2+2A\cos\theta)^3 \quad (m = 0, 1, 2, 3)$$

and these can be written as the sum of terms of the form $\cos n\theta/(1+A^2+2A\cos\theta)^3$ $(n = 0, 1, 2, 3)$ by use of well-known trigonometrical identities.

Hence our basic integral is

$$\begin{aligned} I_n &= \int_0^{2\pi} \frac{\cos n\theta}{(1+A^2+2A\cos\theta)^3} d\theta \\ &= \operatorname{Re}\int_0^{2\pi} \frac{e^{in\theta}}{(1+A\,e^{i\theta})^3(1+A\,e^{-i\theta})^3} d\theta \\ &= \operatorname{Re}\oint_{|z|=1} \frac{-iz^{n+2}}{(1+Az)^3(z+A)^3} dz \\ &= \frac{\pi(-A)^n}{(1-A^2)^5}[(n+2)(n+1)-2A^2(n^2-4)+A^4(n-2)(n-1)],\end{aligned}$$

where we have used Cauchy's theorem of residues and considered only the pole, of order 3, at $z = -A$ in the last integrand. The other pole, at $z = -1/A$, lies outside $|z| = 1$ since A is always less than 1; so

$$\bar{\eta} = -\frac{A\lambda}{\pi^2}\{A(A^4-4A^2+1)I_0+(A^4-3A^2+1)I_1+2A^3I_2+A^2I_3\}-a_0$$

$$= \frac{4A^2\lambda}{\pi(1-A^2)^2}-a_0.$$

Now we are taking the mean level of each wave to be in the plane $y = 0$, hence

$$a_0 = 4A^2\lambda/\pi(1-A^2)^2, \tag{3.11}$$

From (3.9) and (3.10) we obtain

$$\frac{d\eta}{d\hat{x}} = \frac{-4A(1-A^2)\sin 2\pi\alpha}{A^4-6A^2+1+4A^2\cos^2 2\pi\alpha}.$$

Thus

$$1+\left(\frac{d\eta}{d\hat{x}}\right)^2 = \frac{(A^4+2A^2+1-4A^2\cos^2 2\pi\alpha)^2}{(A^4-6A^2+1+4A^2\cos^2 2\pi\alpha)^2} \tag{3.12}$$

and we use this expression to evaluate the potential energy V from (3.3):

$$V = \frac{\tau}{\lambda}\int_0^\lambda\left\{\left[1+\left(\frac{d\eta}{dx}\right)^2\right]^{\frac{1}{2}}-1\right\}dx$$

$$= \frac{-\tau}{\lambda}\int_0^1\left\{\left[1+\left(\frac{d\eta}{d\hat{x}}\right)^2\right]^{\frac{1}{2}}-1\right\}\frac{d\hat{x}}{d\alpha}d\alpha$$

$$= \frac{\tau}{\lambda}\int_0^1\frac{-4A\cos 2\pi\alpha}{(1+A^2+2A\cos 2\pi\alpha)}d\alpha$$

$$= \frac{\tau}{2\pi\lambda}\operatorname{Re}\int_0^{2\pi}\frac{-4A\,e^{i\theta}}{(1+A\,e^{i\theta})(1+A\,e^{-i\theta})}d\theta$$

$$= \frac{\tau}{2\pi\lambda}\operatorname{Re}\oint_{|z|=1}\frac{4iAz}{(1+Az)(z+A)}dz,$$

where the positive root of $[1+(d\eta/d\hat{x})^2]^{\frac{1}{2}}$ is taken, since this corresponds to the physical situation for most values of A. However when $A = A_{crit}$ (A_{crit} being the smallest real root of A^4-6A^2+1), we have an infinite slope in the wave profile at $\alpha = \frac{1}{4}$. For $A_{crit} < A \leqslant A_{max}$ the slope is infinite at two values of α (see figure 4). In fact $A_{crit} = 0{\cdot}414214$, corresponding to a steepness of $0{\cdot}636620$. We shall return to consider this situation towards the end of this section.

To evaluate V, we note that only the simple pole at $z = -A$ is relevant and easily obtain

$$V = \frac{4A^2}{1-A^2}\tau, \tag{3.13}$$

or

$$V = 2[(1+\pi^2a^2/4\lambda^2)^{\frac{1}{2}}-1]\tau. \tag{3.14}$$

Thus V is a strictly increasing function of the steepness a/λ, and for small values of a/λ becomes

$$V = \frac{\pi^2 a^2}{4\lambda^2}\tau + O\left(\frac{a^4}{\lambda^4}\right).$$

This is exactly as predicted by linear theory (cf. Lamb 1932, §266).

To evaluate the kinetic energy we must use (2.10) and (2.11), from which

$$T = \frac{c}{2\lambda}\int_0^\lambda \eta\{c-(1+\eta'^2)^{\frac{1}{2}}[(u_s-c)^2+v_s^2]^{\frac{1}{2}}\}\,dx$$
$$= -\frac{c}{2\lambda}\int_0^\lambda \eta(1+\eta'^2)^{\frac{1}{2}}[(u_s-c)^2+v_s^2]^{\frac{1}{2}}\,dx$$

since the mean level is zero. Now along $\Psi = 0$, with $z = \hat{x}+iy$, we have

$$\left|\frac{d\alpha}{dz}\right| = \frac{1}{c\lambda}(\Phi_{\hat{x}}^2+\Phi_\eta^2)^{\frac{1}{2}} = \frac{1}{c\lambda}[(u_s-c)^2+v_s^2]^{\frac{1}{2}}.$$

We can readily show, using (3.6) and (3.7), that

$$\left|\frac{d\alpha}{dz}\right| = \frac{1+A^2+2A\cos 2\pi\alpha}{\lambda(1+A^2-2A\cos 2\pi\alpha)}.$$

So we have

$$T = \frac{c^2 a_0}{2} - \frac{c^2 A\lambda}{\pi}\int_0^1 \frac{(A+\cos 2\pi\alpha)}{(1+A^2+2A\cos 2\pi\alpha)}\,d\alpha$$

and the integral is easily seen to vanish once it is written in complex form. Hence $T = \frac{1}{2}c^2 a_0$ and, from (2.6), $I = ca_0$.

The square of the phase speed of pure capillary waves is given by

$$c^2 = \frac{2\pi\tau}{\lambda}\frac{(1-A^2)}{(1+A^2)}$$

[Crapper 1957, equation (60)]. Hence, using (3.11),

$$T = \frac{4A^2}{(1-A^4)}\tau, \tag{3.15}$$

or

$$T = \frac{\pi^2 a^2}{4\lambda^2}\left(1+\frac{\pi^2 a^2}{4\lambda^2}\right)^{-\frac{1}{2}}\tau. \tag{3.16}$$

So T is a strictly increasing function of a/λ and for small values of this parameter

$$T = \frac{\pi^2 a^2}{4\lambda^2}\tau + O\left(\frac{a^4}{\lambda^4}\right),$$

which again accords with linear theory.

We see, using (3.13) and (3.15), that

$$V = T(1+A^2).$$

So $V > T$ in general, with $V \doteqdot T$ only for very small steepnesses. From (3.14) and (3.16) we get, finally,

$$V - T = \frac{\pi^4 a^4}{64\lambda^4}\tau + O\left(\frac{a^6}{\lambda^6}\right),$$

$a/2\pi$	T	V	S_{xx}	F
0·1	0·024375	0·024524	0·072977	0·072681
0·2	0·094159	0·096374	0·280261	0·275907
0·3	0·200879	0·210942	0·592575	0·573169
0·4	0·334277	0·362020	0·975088	0·922785
0·5	0·485115	0·543109	1·397352	1·290621
0·6	0·646414	0·748283	1·837374	1·654308
0·7	0·813458	0·972559	2·281275	2·001734
0·729765	0·863816	1·042388	2·412875	2·101116

TABLE 1. Values of T, V, S_{xx} and F for pure capillary waves for particular values of the steepness $a/2\pi$ ($\tau = 1$, $\lambda = 2\pi$).

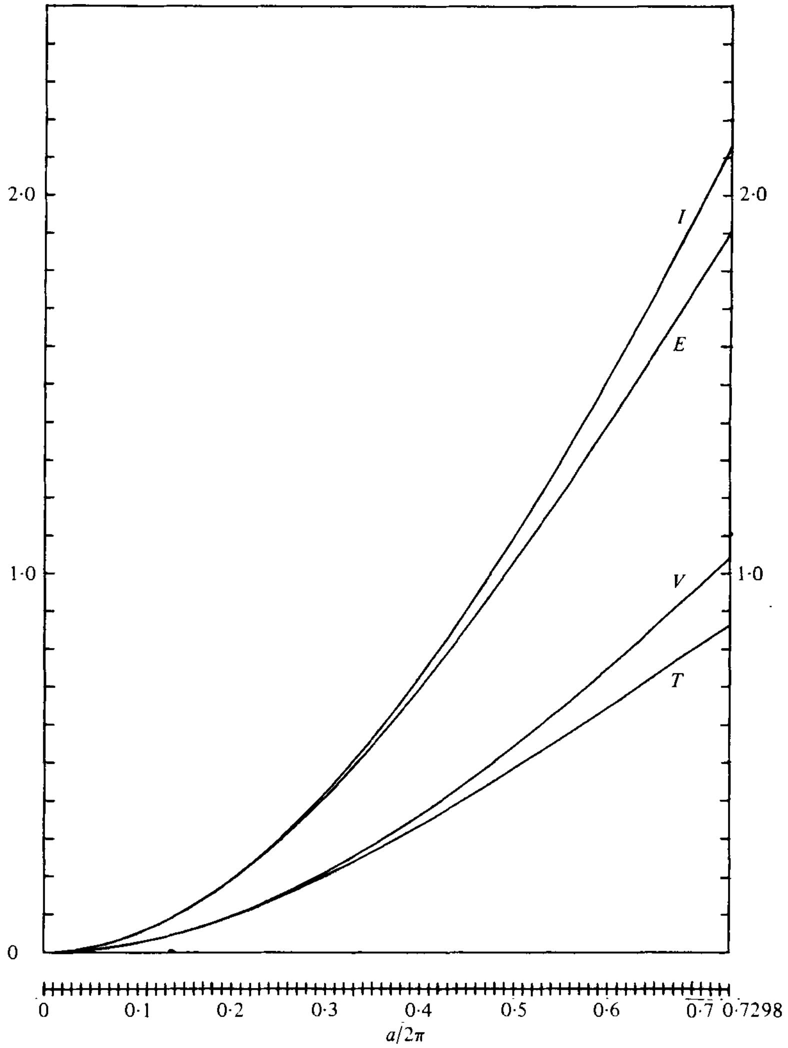

FIGURE 2. Impulse I, total energy E, potential energy V and kinetic energy T as functions of $a/2\pi$ for pure capillary waves.

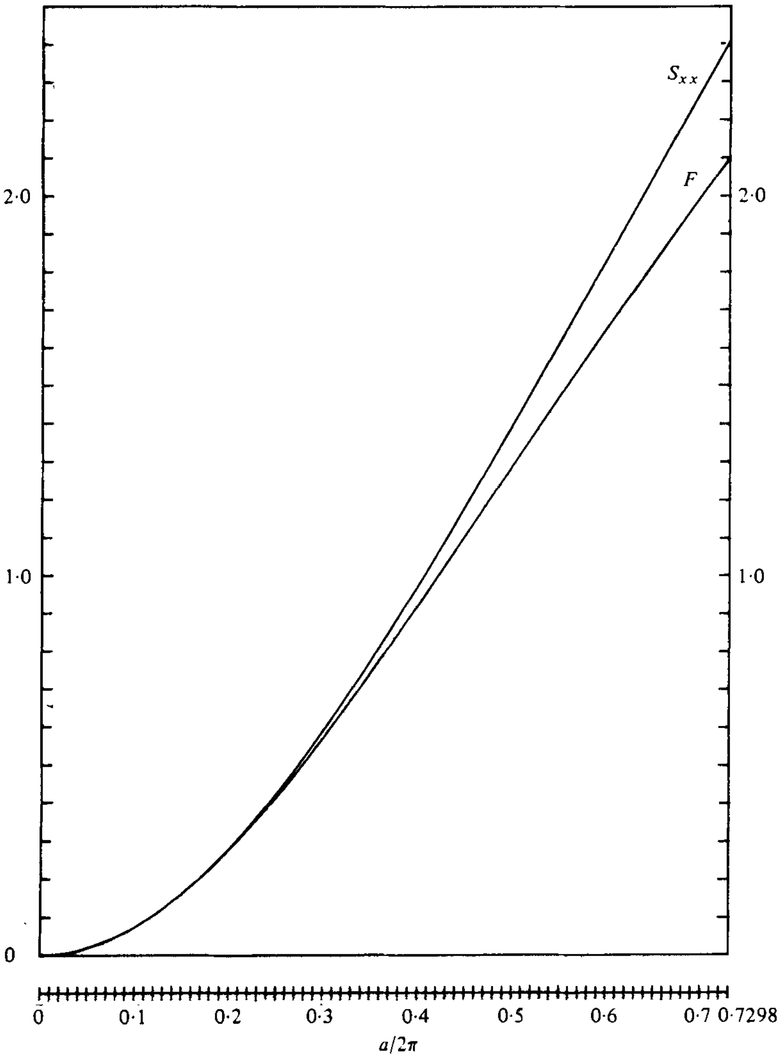

FIGURE 3. Radiation stress S_{xx} and total energy flux F as functions of $a/2\pi$ for pure capillary waves.

again in agreement with linear theory. Equations (3.13) and (3.15) can also be obtained from Lighthill's (1965) analysis.

Evaluation of radiation stresses and energy flux

We now evaluate the radiation stresses and energy flux for deep-water pure capillary waves. Since $g = 0$, we write (2.23) and (2.24) as

$$S_{xx} = 4T - V, \quad F = 3cT. \qquad (3.17), (3.18)$$

Using (3.13) and (3.15), these become

$$S_{xx} = \frac{4A^2(3-A^2)}{(1-A^4)}\tau, \quad F = \frac{12A^2c}{(1-A^4)}\tau. \qquad (3.19), (3.20)$$

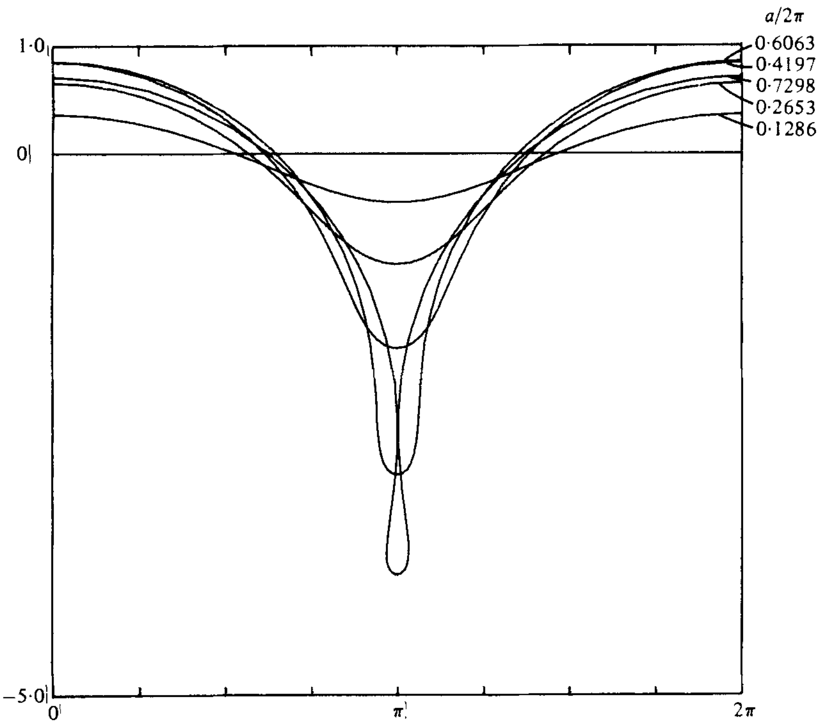

FIGURE 4. Pure capillary wave profiles, all with the same mean level.

For small values of A (and hence a/λ), $S_{xx} \doteqdot 3T \doteqdot \frac{3}{2}E$, and $F \doteqdot Ec_g$ (where $c_g (= \frac{3}{2}c)$ is the group velocity of deep-water capillary waves) in agreement with linear theory. For pure gravity waves, (2.23) and (2.24) become $S_{xx} = 4T - 3V$ and $F = (3T - 2V)c$, as derived by Longuet-Higgins (1975).

Throughout this section we have always taken the positive root of $(1 + \eta'^2)^{\frac{1}{2}}$, as given in (3.12), since for most values of the wave steepness this corresponds to a wave slope of less than 90° (see figure 4). However the results already obtained for V, T, S_{xx} and F are analytic for all $A < A_{\max}$, so by analytic continuation they are valid in the interval $A_{\mathrm{crit}} \leqslant A \leqslant A_{\max}$ despite the fact that the surface bends over itself here.

4. Numerical results

In table 1 we give values for the four quantities V, T, S_{xx} and F [as given by (3.14), (3.16), (3.17) and (3.18) respectively] for particular values of the wave steepness.

In figure 2 we show I, E, V and T *vs.* a/λ, where, for convenience, we have written $\tau = 1$ and $\lambda = 2\pi$. In figure 3 we show S_{xx} and F *vs.* a/λ in a similar manner.

In figure 4 we have drawn particular wave profiles, all with the same mean level. It is clear that the crest height increases and then decreases whereas the trough depth always increases with wave height [this can be shown rigorously, using (3.7) and (3.11)]. This figure is to scale.

I should like to thank Professor M. S. Longuet-Higgins for suggesting this work to me and for his continual guidance and exacting criticisms, and also the Natural Environment Research Council for financial support. Results from this paper were presented in talks given by the author at the 20th British Theoretical Mechanics Colloquium, held at the University of Sheffield on 10–13 April 1978, and at EUROMECH 102, held at the University of Bristol on 18–21 July 1978.

REFERENCES

Barakat, R. & Houston, A. 1968 *J. Geophys. Res.* **73**, 6545–6555.

Beckert, H. 1963 *Arch. Rat. Mech. Anal.* **13**, 15–45.

Cokelet, E. D. 1977 *Phil. Trans. Roy. Soc.* A **286**, 183–230.

Crapper, G. D. 1957 *J. Fluid Mech.* **2**, 532–540.

Harrison, W. J. 1909 *Proc. Lond. Math. Soc.* (2), **7**, 107–121.

Kamesvara Rav, J. C. 1920 *Proc. Ind. Ass. Cultiv. Sci.* **6**, 175–193.

Kinnersley, W. 1976 *J. Fluid Mech.* **77**, 229–241.

Lamb, H. 1932 *Hydrodynamics*, 6th edn. Cambridge University Press.

Lekoudis, S. G., Nayfeh, A. H. & Saric, W. S. 1977 *Phys. Fluids* **20**, 1793–1795.

Levi-Civita, T. 1925 *Math. Ann.* **93**, 264–314.

Lighthill, M. J. 1965 *J. Inst. Math. Appl.* **1**, 269–306.

Longuet-Higgins, M. S. 1975 *Proc. Roy. Soc.* A **342**, 157–174.

Longuet-Higgins, M. S. 1977 *NATO Symp. Turbulent Fluxes, Wave Dyn. Prediction, Île de Bandor.*

McGoldrick, L. F. 1970 *J. Fluid Mech.* **42**, 193–200.

Nayfeh, A. H. 1970*a* *J. Fluid Mech.* **40**, 671–684.

Nayfeh, A. H. 1970*b* *Phys. Fluids* **13**, 545–550.

Nayfeh, A. H. 1971 *J. Fluid Mech.* **48**, 385–395.

Nayfeh, A. H. 1973 *J. Fluid Mech.* **59**, 803–816.

Pierson, W. J. & Fife, P. 1961 *J. Geophys. Res.* **66**, 163–179.

Sekerzh-Zenchovich, Y. I. 1956 *Dokl. Akad. Nauk SSSR* **109**, 913–915.

Slezkin, N. A. 1935 *C.R. Acad. Sci. Paris* **201**, 707–709.

Stokes, G. G. 1847 *Trans. Camb. Phil. Soc.* **8**, 441–455.

Wilton, J. R. 1915 *Phil. Mag.* (6), **29**, 688–700.